RMO–1991

1. Let P be an interior point of a triangle ABC and AP, BP, CP meet the sides BC, CA, AB in D, E, F respectively. Show that

$$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}.$$

2. If a, b, c and d are any 4 positive real numbers, then prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 4.$$

3. A four-digit number has the following properties :

   (a) it is a perfect square,
   (b) its first two digits are equal to each other,
   (c) its last two digits are equal to each other.

   Find all such four-digit numbers.

4. There are two urns each containing an arbitrary number of balls. Both are non-empty to begin with. We are allowed two types of operations:

   (a) remove an equal number of balls simultaneously from the urns, and
   (b) double the number of balls in any one of them.

   Show that after performing these operations finitely many times, both the urns can be made empty.

5. Take any point P_1 on the side BC of a triangle ABC and draw the following chain of lines : P_1P_2 parallel to AC (P_2 on AB), P_2P_3 parallel to BC, P_3P_4 parallel to AB, P_4P_5 parallel to CA, and P_5P_6 parallel to BC. Here P_2, P_5 lie on AB; P_3, P_6 lie on CA; and P_4 on BC. Show that P_6P_1 is parallel to AB.

6. Find all integer values of a such that the quadratic expression

$$(x + a)(x + 1991) + 1$$

   can be factored as a product $(x + b)(x + c)$ where b and c are integers.

7. Prove that $n^4 + 4^n$ is composite for all integer values of n greater than 1.

8. The 64 squares of an 8×8 chessboard are filled with positive integers in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or a common vertex. Thus a square can have 8, 5 or 3 neighbours depending on its position). Show that all the 64 integer entries are in fact equal.