RMO–2006

1. Let ABC be an acute-angled triangle and let D, E, F be the feet of perpendiculars from A, B, C respectively to BC, CA, AB. Let the perpendiculars from F to CB, CA, AD, BE meet them in P, Q, M, N respectively. Prove that P, Q, M, N are *collinear*.

2. Find the least possible value of $a + b$, where a, b are positive integers such that 11 divides $a + 13b$ and 13 divides $a + 11b$.

3. If a, b, c are three positive real numbers, prove that

$$\frac{a^2+1}{b+c} + \frac{b^2+1}{c+a} + \frac{c^2+1}{a+b} \geq 3.$$

4. A 6×6 square is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have only integer sides. Prove that there are always **two** congruent rectangles.

5. Let $ABCD$ be a quadrilateral in which AB is parallel to CD and perpendicular to AD; $AB = 3CD$; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.

6. Prove that there are infinitely many positive integers n such that $n(n+1)$ can be expressed as a sum of two positive squares in *at least* two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.)

7. Let X be the set of all positive integers greater than or equal to 8 and let $f : X \to X$ be a function such that $f(x + y) = f(xy)$ for all $x \geq 4$, $y \geq 4$. If $f(8) = 9$, determine $f(9)$.