RMO–2004

1. Consider in the plane a circle Γ with center O and a line l not intersecting circle Γ. Prove that there is a point Q on the perpendicular drawn from O to the line l, such that for any point P on the line l, PQ represents the length of the tangent from P to the circle Γ.
2. Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T.
3. Let α and β be the roots of the quadratic equation $x^2 + mx - 1 = 0$, where m is an odd integer. Let $\lambda_n = \alpha^n + \beta^n$, for $n \geq 0$. Prove that for $n \geq 0$,

   (a) λ_n is an integer; and

   (b) $gcd(\lambda_n, \lambda_{n+1}) = 1$.
4. Prove that the number of triples (A, B, C) where A, B, C are subsets of $\{1, 2, \cdots, n\}$ such that $A \cap B \cap C = \phi, A \cap B \neq \phi, B \cap C \neq \phi$ is $7^n - 2.6^n + 5^n$.
5. Let $ABCD$ be a quadrilateral ; X and Y be the midpoints of AC and BD respectively ; and the lines through X and Y respectively parallel to BD, AC meet in O. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively. Prove that

   (a) quadrilaterals $APOS$ and $APXS$ have the same area ;

   (b) the areas of the quadrilaterals $APOS, BQOP, CROQ, DSOR$ are all equal .
6. Let $(p_1, p_2, p_3, \cdots, p_n, \cdots)$ be a sequence of primes defined by $p_1 = 2$ and for $n \geq 1, p_{n+1}$ is the largest prime factor of $p_1 p_2 \cdots p_n + 1$. (Thus $p_2 = 3, p_3 = 7$). Prove that $p_n \neq 5$ for any n.
7. Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that

   (a) $y < x < 1$; and

   (b) $x^2 + y^2 < 1$.