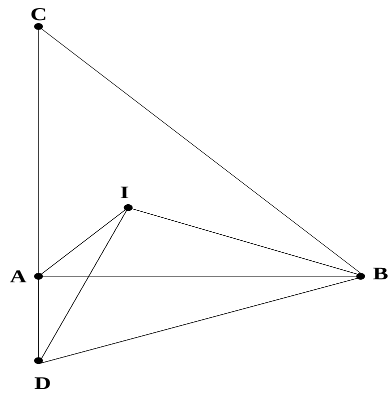
Regional Mathematical Olympiad-2009

Problems and Solutions

1. Let ABC be a triangle in which $AB = AC$ and let I be its in-centre. Suppose $BC = AB + AI$. Find $\angle BAC$.

   Solution:

   We observe that $\angle AIB = 90° + (C/2)$. Extend CA to D such that $AD = AI$. Then $CD = CB$ by the hypothesis. Hence $\angle CDB = \angle CBD = 90° - (C/2)$. Thus

   $$\angle AIB + \angle ADB = 90° + (C/2) + 90° - (C/2) = 180°.$$

   Hence $ADBI$ is a cyclic quadrilateral. This implies that

   $$\angle ADI = \angle ABI = \frac{B}{2}.$$

   But ADI is isosceles, since $AD = AI$. This gives

   $$\angle DAI = 180° - 2(\angle ADI) = 180° - B.$$

   Thus $\angle CAI = B$ and this gives $A = 2B$. Since $C = B$, we obtain $4B = 180°$ and hence $B = 45°$. We thus get $A = 2B = 90°$.

2. Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289.

   Solution: We write

   $$a^2 - 3a - 19 = a^2 - 3a - 70 + 51 = (a - 10)(a + 7) + 51.$$

   Suppose 289 divides $a^2 - 3a - 19$ for some integer a. Then 17 divides it and hence 17 divides $(a - 10)(a + 7)$. Since 17 is a prime, it must divide $(a - 10)$ or $(a + 7)$. But $(a+7)-(a-10) = 17$. Hence whenever 17 divides one of $(a-10)$ and $(a+7)$, it must divide the other also. Thus $17^2 = 289$ divides $(a - 10)(a + 7)$. It follows that 289 divides 51, which is impossible. Thus, there is no integer a for which 289 divides $a^2 - 3a - 19$.

3. Show that $3^{2008} + 4^{2009}$ can be wriiten as product of two positive integers each of which is larger than 2009^{182}.

Solution: We use the standard factorisation:

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2).$$

We observe that for any integers x, y,

$$x^2 + 2xy + 2y^2 = (x + y)^2 + y^2 \geq y^2,$$

and

$$x^2 - 2xy + 2y^2 = (x - y)^2 + y^2 \geq y^2.$$

We write

$$3^{2008} + 4^{2009} = 3^{2008} + 4\big(4^{2008}\big) = \big(3^{502}\big)^4 + 4\big(4^{502}\big)^4.$$

Taking $x = 3^{502}$ and $y = 4^{502}$, we se that $3^{2008} + 4^{2009} = ab$, where

$$a \geq \big(4^{502}\big)^2, \quad b \geq \big(4^{502}\big)^2.$$

But we have

$$\big(4^{502}\big)^2 = 2^{2008} > 2^{2002} = \big(2^{11}\big)^{182} > (2009)^{182},$$

since $2^{11} = 2048 > 2009$.

4. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.

Solution: Let X denote the set of all 3-digit natural numbers; let O be those numbers in X having only *odd* digits; and E be those numbers in X having only *even* digits. Then $X \setminus (O \cup E)$ is the set of all 3-digit natural numbers having at least one odd digit and at least one even digit.The desired sum is therefore

$$\sum_{x \in X} x - \sum_{y \in O} y - \sum_{z \in E} z.$$

It is easy to compute the first sum;

$$\begin{aligned} \sum_{x \in X} x &= \sum_{j=1}^{999} j - \sum_{k=1}^{99} k \\ &= \frac{999 \times 1000}{2} - \frac{99 \times 100}{2} \\ &= 50 \times 9891 = 494550. \end{aligned}$$

Consider the set O. Each number in O has its digits from the set $\{1, 3, 5, 7, 9\}$. Suppose the digit in unit's place is 1. We can fill the digit in ten's place in 5 ways and the digit in hundred's place in 5 ways. Thus there are 25 numbers in the set O each of which has 1 in its unit's place. Similarly, there are 25 numbers whose digit in unit's place is 3; 25 having its digit in unit's place as 5; 25 with 7 and 25 with 9. Thus the sum of the digits in unit's place of all the numbers in O is

$$25(1 + 3 + 5 + 7 + 9) = 25 \times 25 = 625.$$

A similar argument shows that the sum of digits in ten's place of all the numbers in O is 625 and that in hundred's place is also 625. Thus the sum of all the numbers in O is

$$625(10^2 + 10 + 1) = 625 \times 111 = 69375.$$

Consider the set E. The digits of numbers in E are from the set $\{0, 2, 4, 6, 8\}$, but the digit in hundred's place is never 0. Suppose the digit in unit's place is 0. There are $4 \times 5 = 20$ such numbers. Similarly, 20 numbers each having digits 2,4,6,8 in their unit's place. Thus the sum of the digits in unit's place of all the numbers in E is

$$20(0 + 2 + 4 + 6 + 8) = 20 \times 20 = 400.$$

A similar reasoning shows that the sum of the digits in ten's place of all the numbers in E is 400, but the sum of the digits in hundred's place of all the numbers in E is $25 \times 20 = 500$. Thus the sum of all the numbers in E is

$$500 \times 10^2 + 400 \times 10 + 400 = 54400.$$

The required sum is

$$494550 - 69375 - 54400 = 370775.$$

5. A convex polygon Γ is such that the distance between any two vertices of Γ does not exceed 1.

   (i) Prove that the distance between any two points on the boundary of Γ does not exceed 1.

   (ii) If X and Y are two distinct points inside Γ, prove that there exists a point Z on the boundary of Γ such that $XZ + YZ \leq 1$.

Solution:

(i) Let S and T be two points on the boundary of Γ, with S lying on the side AB and T lying on the side PQ of Γ. (See Fig. 1.) Join TA, TB, TS. Now ST lies between TA and TB in triangle TAB. One of $\angle AST$ and $\angle BST$ is at least 90°, say $\angle AST \geq 90°$. Hence $AT \geq TS$. But AT lies inside triangle APQ and one of $\angle ATP$ and $\angle ATQ$ is at least 90°, say $\angle ATP \geq 90°$. Then $AP \geq AT$. Thus we get $TS \leq AT \leq AP \leq 1$.

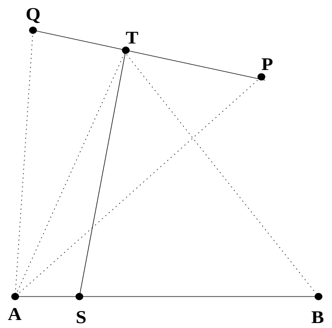


Fig. 1

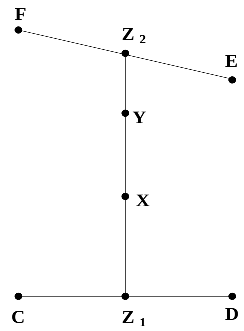


Fig. 2

(ii) Let X and Y be points in the interior Γ. Join XY and produce them on either side to meet the sides CD and EF of Γ at Z_1 and Z_2 respectively. WE have

$$\begin{aligned}(XZ_1 + YZ_1) + (XZ_2 + YZ_2) &= (XZ_1 + XZ_2) + (YZ_1 + YZ_2) \\ &= 2Z_1Z_2 \leq 2,\end{aligned}$$

by the first part. Therefore one of the sums $XZ_1 + YZ_1$ and $XZ_2 + YZ_2$ is at most 1. We may choose Z accordingly as Z_1 or Z_2.

6. In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

Solution: Suppose r pages of the book are torn off. Note that the page numbers on both the sides of a page are of the form $2k-1$ and $2k$, and their sum is $4k-1$. The sum of the numbers on the torn pages must be of the form

$$4k_1 - 1 + 4k_2 - 1 + \cdots + 4k_r - 1 = 4(k_1 + k_2 + \cdots + k_r) - r.$$

The sum of the numbers of all the pages in the untorn book is

$$1 + 2 + 3 + \cdots + 100 = 5050.$$

Hence the sum of the numbers on the torn pages is

$$5050 - 4949 = 101.$$

We therefore have

$$4(k_1 + k_2 + \cdots + k_r) - r = 101.$$

This shows that $r \equiv 3 \pmod 4$. Thus $r = 4l + 3$ for some $l \geq 0$.

Suppose $r \geq 7$, and suppose $k_1 < k_2 < k_3 < \cdots < k_r$. Then we see that

$$\begin{aligned}4(k_1 + k_2 + \cdots + k_r) - r &\geq 4(k_1 + k_2 + \cdots + k_7) - 7 \\ &\geq 4(1 + 2 + \cdots + 7) - 7 \\ &= 4 \times 28 - 7 = 105 > 101.\end{aligned}$$

Hence $r = 3$. This leads to $k_1 + k_2 + k_3 = 26$ and one can choose distinct positive integers k_1, k_2, k_3 in several ways.

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