Problem of the week

The side lengths of a triangle are three consecutive positive integers and the largest angle in the triangle is twice the smallest one. Determine the side lengths of the triangle.

Solution:

Suppose that the side lengths of the triangle are $n-1$, n, and $n+1$ for some positive integer $n \geq 3$. (Smaller values of n do not give a triangle.) Suppose that the smallest angle is θ, which is opposite the shortest side (of length $n-1$). Thus, the largest angle is 2θ, which is opposite the longest side (of length $n+1$).

By the Law of Sines, $\frac{n-1}{\sin\theta} = \frac{n+1}{\sin 2\theta}$ and so $\frac{n-1}{\sin\theta} = \frac{n+1}{2\sin\theta\cos\theta}$. Since $\sin\theta \neq 0$, then $\cos\theta = \frac{n+1}{2(n-1)}$.

Now, by the Law of Cosines,

$$\begin{aligned}
(n-1)^2 &= n^2 + (n+1)^2 - 2n(n+1)\cos\theta\,; \\
(n-1)^2 &= n^2 + (n+1)^2 - 2n(n+1)\frac{n+1}{2(n-1)} \\
n^2 - 2n + 1 &= n^2 + n^2 + 2n + 1 - \frac{n(n+1)^2}{n-1}\,; \\
\frac{n(n+1)^2}{n-1} &= n^2 + 4n\,; \\
n(n^2+2n+1) &= (n^2+4n)(n-1)\,; \\
n^2 + 2n + 1 &= (n+4)(n-1) \qquad (\text{since } n \neq 0)\,; \\
n^2 + 2n + 1 &= n^2 + 3n - 4\,; \\
n &= 5\,.
\end{aligned}$$

Therefore, $n = 5$, and the side lengths are 4, 5, and 6.