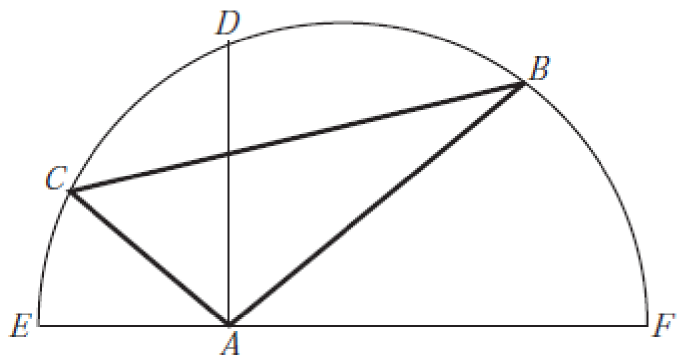
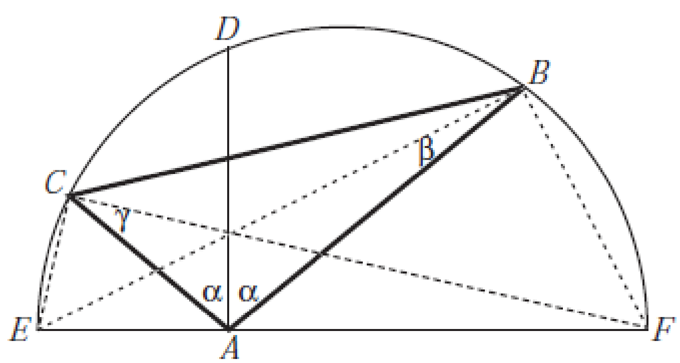
Problem of the week

Given a semicircle with diameter EF as indicated, triangle ABC with A lying on the diameter, and B and C on the semicircle. AD bisects $\angle BAC$, BE bisects $\angle ABC$, and CF bisects $\angle ACB$. If AB = 6, AC = 3, BC = 7, and $AD \perp EF$, find AD.



Solu:

Any triangle inscribed in a semicircle with the diameter as one side is a *right* triangle. So _EDF is a right triangle, and, as in the ciphering problem number 3, we have $\boldsymbol{AD^2}$ = (AE)(AF).



Similarly, $\triangle ECF$ and $\triangle EBF$ are right triangles. Note also that $2\alpha + 2\beta + 2\gamma = \pi$, so $\alpha + \beta + \gamma = \pi/2$.

Next, consider $\triangle AEC$ and $\triangle AFB$. $\angle CAE = \angle BAF = \pi/2 - \alpha$. $\angle ABF = \pi/2 - \beta$ and $\angle ACE = \pi/2 - \gamma$. Thus,

$$\angle ACE = \pi/2 - \gamma = \alpha + \beta = \pi - (\pi/2 - \alpha) - (\pi/2 - \beta) = \angle AFB.$$

Thus, $\triangle ACE \sim \triangle AFB$, and so $\dfrac{AE}{AC} = \dfrac{AB}{AF}$. It follows that

$$(AD)^2 = (AE)(AF) = (AB)(AC) = 18,$$

so $AD = 3\sqrt{2}$.