INMO–1991

Time : 4 hours

Attempt as many questions as you possibly can

Use of calculating aids not permitted

1. Find the number of positive integers n for which

   (a) $n \leq 1991$ and

   (b) 6 is a factor of $(n^2 + 3n + 2)$.

2. Given any acute-angled triangle ABC, let points A', B', C' be located as follows : A' is the point where altitude from A on BC meets the outwards facing semi-circle drawn on BC as diameter. Points B', C' are located similarly. Prove that

$$[BCA']^2 + [CAB']^2 + [ABC']^2 = [ABC]^2,$$

   where $[ABC]$ denotes the area of triangle ABC, etc.

3. Given a triangle ABC, define the quantities x, y, z as follows:

$$\begin{aligned} x &= \tan((B-C)/2)\tan(A/2) \\ y &= \tan((C-A)/2)\tan(B/2) \\ z &= \tan((A-B)/2)\tan(C/2). \end{aligned}$$

   Prove that : $x + y + z + xyz = 0$.

4. Let a, b, c be real numbers with $0 < a < 1$, $0 < b < 1$, $0 < c < 1$ and $a + b + c = 2$. Prove that :

$$\frac{a}{1-a} \cdot \frac{b}{1-b} \cdot \frac{c}{1-c} \geq 8.$$

5. Triangle ABC has incenter I. Let points X, Y be located on the line segment AB, AC respectively so that :

$$BX \cdot AB = IB^2 \text{ and } CY \cdot AC = IC^2$$

   Given that the points X, I, Y lie on a straight line, find the possible values of the measure of angle A.

6. (a) Determine the set of all positive integers n for which

$$3^{n+1} \text{ divides } 2^{j^n} + 1$$

   (b) Prove that 3^{n+2} does not divide $2^{3^n} + 1$ for any positive integer n.

7. Solve the following system of equations for real x, y, z:

$$\begin{aligned} x + y - z &= 4 \\ x^2 - y^2 + z^2 &= 4 \\ xyz &= 6 \end{aligned}$$

8. There are 10 objects with total weight 20, each of the weights being a positive integer. Given that none of the weights exceeds 10, prove that the 10 objects can be divided into two groups that balance each other when placed on the two pans of a balance.

9. Triangle ABC has incenter I, its incircle touches the side BC at T. The line through T parallel to IA meets the incircle again at S and the tangent to the incircle at S meets the sides AB, AC at points C', B' respectively. Prove that the triangle AB'C' is similar to triangle ABC.

10. For any positive integer n, let $S(n)$ denote the number of ordered pairs (x, y) of positive integers for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

(for instance, $S(2) = 3$). Determine the set of positive integers n for which $S(n) = 5$.