INMO–2006

1. In a nonequilateral triangle ABC, the sides a, b, c form an arithmetic progression. Let I and O denote the incentre and circumcentre of the triangle respectively.

   (i) Prove that IO is perpendicular to BI.

   (ii) Suppose BI extended meets AC in K, and D, E are the mid-points of BC, BA respectively. Prove that I is the circumcentre of triangle DKE.

2. Prove that for every positive integer n there exists a **unique** ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a+b-1)(a+b-2) + a.$$

3. Let X denote the set of all triples (a, b, c) of integers. Define a function $f : X \to X$ by

$$f(a, b, c) = (a+b+c, ab+bc+ca, abc).$$

   Find all triples (a, b, c) in X such that $f(f(a, b, c)) = (a, b, c)$.

4. Some 46 squares are randomly chosen from a 9×9 chess board and are coloured red. Show that there exists a 2×2 block of 4 squares of which at least three are coloured red.

5. In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120^\circ$, and $\angle ABD = 30^\circ$. Prove that

   (i) $c \geq a + b$;

   (ii) $|\sqrt{c+a} - \sqrt{c+b}| = \sqrt{c-a-b}$.

6. (a) Prove that if n is a positive integer such that $n \geq 4011^2$, then there exists an integer l such that $n < l^2 < (1 + \frac{1}{2005})n$.

   (b) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$, there exists an integer l, such that $n < l^2 < (1 + \frac{1}{2005})n$.