INMO–2005

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1. Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect the incircle of ABC at K and L, K being nearer to A than L. If $AK = KL = LM$, prove that the sides of triangle ABC are in the ratio $5 : 10 : 13$ in some order.

2. Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}$$

.

   Find the minimum possible value of β.

3. Let p, q, r be positive real numbers, not all equal, such that some two of the equations

$$px^2 + 2qx + r = 0, qx^2 + 2rx + p = 0, rx^2 + 2px + q = 0,$$

   have a common root, say α. Prove that

   (a) α is real and negative; and

   (b) the remaining third equation has non-real roots.

4. All possible 6-digit numbers, in each of which the digits occur in non-increasing order (from left to right, e.g.,877550) are written as a sequence in increasing order. Find the 2005-th number in this sequence.

5. Let x_1 be a given positive integer. A sequence$(x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, \cdots)$ of positive integers is such that x_n, for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1}. Prove that

   (a) the sequence has an even number;

   (b) the sequence has infinitely many even numbers.

6. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x^2 + yf(z)) = xf(x) + zf(y),$$

   for all x, y, z in $\mathbf{R}$. (Here $\mathbf{R}$ denotes the set of all real numbers.)