

# INMO–2004

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1. Consider a convex quadrilateral  $ABCD$ , in which  $K, L, M, N$  are the midpoints of the sides  $AB, BC, CD, DA$  respectively . Suppose
  - (a)  $BD$  bisects  $KM$  at  $Q$  ;
  - (b)  $QA = QB = QC = QD$  ; and
  - (c)  $LK/LM = CD/CB$  .

Prove that  $ABCD$  is a square .

2. Suppose  $p$  is a prime greater than 3. Find all pairs of integers  $(a, b)$  satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

3. If  $\alpha$  is a real root of the equation  $x^5 - x^3 + x - 2 = 0$  , prove that  $[\alpha^6] = 3$  . (For any real number  $a$  , we denote by  $[a]$  the greatest integer not exceeding  $a$  . )
4. Let  $R$  denote the circumradius of a triangle  $ABC$ ;  $a, b, c$  its sides  $BC, CA, AB$  ; and  $r_a, r_b, r_c$  its exradii opposite  $A, B, C$ . If  $2R \leq r_a$  , prove that
  - (i)  $a > b$  and  $a > c$  ;
  - (ii)  $2R > r_b$  and  $2R > r_c$  .
5. Let  $S$  denote the set of all 6-tuples  $(a, b, c, d, e, f)$  of positive integers such that  $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$ . Consider the set

$$T = \{abcdef : (a, b, c, d, e, f) \in S\}.$$

Find the greatest common divisor of all the members of  $T$ .

6. Prove that the number of 5-tuples of positive integers  $(a, b, c, d, e)$  satisfying the equation

$$abcde = 5(bcde + acde + abde + abce + abcd)$$

is an odd integer .