

# INMO–2003

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1. Consider an acute triangle  $ABC$  and let  $P$  be an interior point of  $ABC$ . Suppose the lines  $BP$  and  $CP$ , when produced, meet  $AC$  and  $AB$  in  $E$  and  $F$  respectively. Let  $D$  be the point where  $AP$  intersects the line segment  $EF$  and  $K$  be the foot of perpendicular from  $D$  on to  $BC$ . Show that  $DK$  bisects  $\angle EKF$ .
2. Find all primes  $p$  and  $q$ , and even numbers  $n > 2$ , satisfying the equation

$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$

3. Show that for every real number  $a$  the equation

$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0$$

has at least one non-real root and find the sum of all the non-real roots of the equation.

4. Find all 7-digit numbers formed by using only the digits 5 and 7, and divisible by both 5 and 7.
5. Let  $ABC$  be a triangle with sides  $a, b, c$ . Consider a triangle  $A_1B_1C_1$  with sides equal to  $a + \frac{b}{2}, b + \frac{c}{2}, c + \frac{a}{2}$ . Show that

$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$

where  $[XYZ]$  denotes the area of the triangle  $XYZ$ .

6. In a lottery tickets are given nine-digit numbers using only the digits 1, 2, 3. They are also coloured red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours. Suppose the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green. Determine, with proof, the colour of the ticket bearing the number 123123123.