INMO–2002

February 3, 2002

1. For a convex hexagon $ABCDEF$, consider the following six statements :

$$\begin{array}{llll} (a_1) & ABisparalleltoDE: & (a_2) & AE = BD; \\ (b_1) & BCisparalleltoEF: & (b_2) & BF = CE; \\ (c_1) & CDisparalleltoFA: & (c_2) & CA = DF. \end{array}$$

   (a) Show that if all the six statements are true , then the hexagon is cyclic (i. e. , it can be inscribed in a circle) .

   (b) Prove that , in fact , any five of these six statements also imply that the hexagon is cyclic .

2. Determine the least positive value taken by the expression $a^3 + b^3 + c^3 = 3abc$ as a, b, c vary over all positive integers . Find also all triples (a, b, c) for which this least value is attained .

3. Let x, y be positive reals such that $x + y = 2$. Prove that

$$x^3y^3(x^3 + y^3) \leq 2.$$

4. Do there exist 100 lines in the plane , no three of them concurrent , such that they intersect exactly in 2002 points ?

5. Do there exist three distinct positive real numbers a, b, c such that the numbers $a, b, c, b+c-a, c+a-b, a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order ?

6. Suppose the n^2 numbers $1, 2, 3, \cdots, n^2$ are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row (from left to right) and each column (from top to bottom) are in increasing order . Denote by a_{jk} the number in the $j-th$ row and $k-th$ column . Suppose b_j is the maximum possible number of entries that can occur as $a_{jj}, 1 \leq j \leq n$. Prove that

$$b_1 + b_2 + b_3 + \cdots b_n \leq \frac{n}{3}(n^2 - 3n + 5).$$

   (Example : In the case $n = 3$, the only numbers which can occur as a_{22} are $4, 5$ or 6 so that $b_2 = 3$.)