INMO–2001

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1. Let ABC be a triangle in which *no* angle is $90°$. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements :

   (a) If P is the incentre or an excentre of ABC, then P is the circumcentre of $A_1B_1C_1$;

   (b) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$;

   (c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.

2. Show that the equation
$$x^2 + y^2 + z^2 = (x - y)(y - z)(z - x)$$
has infinitely many solutions in integers x, y, z.

3. If a, b, c are positive real numbers such that $abc = 1$, prove that
$$a^{b+c}b^{c+a}c^{a+b} \leq 1.$$

4. Given any nine integers show that it is possible to choose, from among them, four integers a, b, c, d such that $a + b - c - d$ is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

5. Let ABC be a triangle and D be the mid-point of side BC. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15°$. Show that $\angle ADC$ is obtuse. Further, if O is the circumcentre of ADC, prove that triangle AOD is equilateral.

6. Let $\mathcal{R}$ denote the set of real numbers. Find all functions $f : \mathcal{R} \to \mathcal{R}$ satisfying the condition
$$f(x + y) = f(x)f(y)f(xy)$$
for all x, y in $\mathcal{R}$.