INMO–1996

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1. (a) Given any positive integer n, show that there exist distinct positive integers x and y such that $x + j$ divides $y + j$ for $j = 1, 2, 3, \cdots, n$.

   (b) If for some positive integers x and $y, x + j$ divides $y + j$ for all positive integers j, prove that $x = y$.

2. Let C_1 and C_2 be two concentric circles in the plane with radii R and $3R$ respectively. Show that the orthocentre of any triangle inscribed in circle C_1 lies in the interior of circle C_2. Conversely, show that also every point in the interior of C_2 is the orthocentre of some triangle inscribed in C_1.

3. Solve the following system of equations for real numbers a, b, c, d, e.

$$\begin{aligned} 3a &= (b+c+d)^3, \\ 3b &= (c+d+e)^3, \\ 3c &= (d+e+a)^3, \\ 3d &= (e+a+b)^3, \\ 3e &= (a+b+c)^3. \end{aligned}$$

4. Let X be a set containing n elements. Find the number of all ordered triples (A, B, C) of subsets of X such that A is a subset of B and B is a *proper* subset of C.

5. Define a sequence $(a_n)_{n \geq 1}$ by $a_1 = 1$, $a_2 = 2$ and $a_{n+2} = 2a_{n+1} - a_n + 2$ for $n \geq 1$. Prove that for any m, $a_m a_{m+1}$ is also a term in the sequence.

6. There is a $2n \times 2n$ array (matrix) consisting of 0's and 1's and there are exactly $3n$ zeros. Show that it is possible to remove all the zeros by deleting some n rows and some n columns.

   [Note: A $m \times n$ array is a rectangular arrangement of mn numbers in which there are m horizontal rows and n vertical columns.]