INMO–1995

Attempt all questions.

Do not use mathematical tables or calculators.

1. In an acute-angled triangle ABC, $\angle A = 30^o$, H is the orthocenter and M is the mid-point of BC. On the line HM, take a point T such that $HM = MT$. Show that $AT = 2BC$.
2. Show that there are infinitely many pairs (a, b) of relatively prime integers (not necessarily positive) such that both quadratic functions

$$\begin{aligned} x^2 + ax + b &= 0 \\ \text{and } x^2 + 2ax + b &= 0 \end{aligned}$$

   have integer roots.
3. Show that the number of 3−element subset $\{a, b, c\}$ of $\{1, 2, 3, \ldots, 63\}$ with $a + b + c < 95$ is less than the number of those with $a + b + c > 95$.
4. Let ABC be triangle and a circle Γ' be drawn inside the triangle, touching its incircle Γ externally and also touching the two sides AB and AC. Show the ratio of the radii of the circles Γ' and Γ is equal to

$$\tan^2\left(\frac{\pi - A}{4}\right).$$

5. Let $a_1, a_2, a_3, \ldots, a_n$ be n real numbers all greater than 1 and such that $|a_k - a_{k+1}| < 1$ for $1 \leq k \leq n - 1$. Show that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_3}{a_4} + \ldots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1} < 2n - 1.$$

6. Find all primes p for which the quotient

$$(2^{p-1} - 1)$$

   is a square.